Math 347: Homework 9
Due on: Dec. 5, 2018

1. Let p, q and r be three natural numbers that are pairwise relatively prime. Prove that for any a, b, c integer numbers the equations

$$x \equiv a \bmod p$$
$$x \equiv b \bmod q$$
$$x \equiv c \bmod r$$

   have an unique solution module $N = pqr$.

   Apply this result to the problem mentioned in class on Monday Nov. 26. That is: What is the smallest number n whose remainder on division by 3, 5 and 7 is 1, 2 and 4, respectively?

2. For the examples below determine if R is an equivalence relation on S.
   (i) $S = \mathbb{Z}_{\geq 2}$; $(x, y) \in R$ if $\gcd(x, y) > 1$;
   (ii) $S = \mathbb{R}$; $(x, y) \in R$ if and only if there exists $n \in \mathbb{Z}$ such that $x = 2^n y$;

3. Prove that every year (including leap years) has at least one Friday the 13th. What is the maximum number of Friday the 13th in a year?

4. Fermat's little theorem states that if p is prime and a is not a multiple of p, then

$$a^{p-1} \equiv 1 \bmod p.$$

   By Theorem 6.21 in the book, for a and p relatively prime the numbers in the set $\{a, 2a, \ldots, (p-1)a\}$ all have distinct remainder upon division by p. Use this fact to give a proof of Fermat's little theorem.

5. Define f and g from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$ by

$$f(x) = x + a, \quad \text{and} \quad g(x) = ax.$$

   (i) Give a complete description of the functional digraph of f and g;
   (ii) Draw the functional digraph of g for the case $(n, a) = (11, 4)$ and the case $(n, a) = (12, 4)$. Describe a property of the digraph that is true whenever n is prime and false whenver n is not prime.